**Binomial Option Pricing Model**

**Single Period Binomial Trees**

# Single period Binomial Trees

## Forward (Standard) Binomial Tree

* 
* Suppose the stock price **grows at the risk-free rate (Less any dividends)**
* However, the growth this way is not guaranteed – there is **some risk involved**
* We account for this risk by multiplying the growth factor by the **standard deviation**
  + The **up factor** should have a **higher growth** thus **adds the SD over the period**
  + The **down factor** should have a **lower growth** thus **subtracts the SD over the period**
* Known as a Forward Binomial Tree because if the **SD is 0**, the **future stock price becomes the forward price**
* Note that we can only use this method if it was explicitly stated that it is a **Forward Tree**

$u = e^{(r-q)t + \sigma\sqrt{t}}$

$d = e^{(r-q)t - \sigma\sqrt{t}}$





* 
* 

$$u = \frac{S\_u}{S\_0}$$

$$d = \frac{S\_d}{S\_0}$$





## Understanding the Forward Tree

* As the name suggests, it is a Binomial Tree which has a **Forward as the underlying**
* The price of the forward can either increase or decrease each period, based on its standard deviation

$$u\_F = e^{\sigma\sqrt{t}}$$

$$u\_D = e^{-\sigma\sqrt{t}}$$





* Thus, in the following period we will have the following prices:

$$F\_U = F\_0 \cdot e^{\sigma\sqrt{t}}$$

$$F\_D = f\_0 \cdot e^{-\sigma\sqrt{t}}$$





* We assume that the Forward is fairly priced, thus we obtain

$$F\_U = S\_0 \cdot e^{(r-q)t + \sigma\sqrt{t}} = S\_0 \cdot d$$

$$F\_D = S\_0 \cdot e^{(r-q)t - \sigma\sqrt{t}} = S\_0 \cdot u$$





# Replicating Portfolio Method

# Risk Neutral Valuation Method

* In general, this method is **faster than the replicating portfolio (Preferred)**



$$p = \frac{e^{(r-\delta)t} – e^{(r-\delta)-\sigma\sqrt{t}}}{e^{(r-\delta)+\sigma\sqrt{t}}-e^{(r-\delta)-\sigma\sqrt{t}}}$$

$$p = \frac{1-e^{-\sigma\sqrt{t}}}{e^{\sigma\sqrt{t}} – e^{-\sigma\sqrt{t}}}$$

$$p = \frac{1-e^{-\sigma\sqrt{t}}}{(1-e^{-\sigma\sqrt{t}})(1 + e^{\sigma\sqrt{t}})}$$

$$p = \frac{1}{1+e^{\sigma\sqrt{t}}}$$









* Note that this means that the **risk free rate is not needed** to compute the probability

**Multi Period Binomial Trees**

# American Options

* They have the option to be exercised early, thus we need to consider at each intermediate node if the Option should be **exercised immediately or held off** till later
  + **Immediate Exercise Value** → Payoff of the Option at that node
  + **Pull Back Value** → PV of future payoffs
* We compare the two values to determine what should be done
  + **Immediate Exercise Value > Pull Back Value** → Exercise Immediately
  + **Immediate Exercise Value < Pull Back Value** → Exercise in future
  + **Immediate Exercise Value = Pull Back Value** → Indifferent
* 
  + Since the value at each node is not confirmed, there is **no direct method** to calculate the Option Price in one step
  + Since an American Call *without dividends* will never be exercised early, it is identical to a European Call thus the **direct method can be used**

## Solving for Strike Prices

* There are several questions that will ask to solve for the **minimum or maximum strike** that will result in the option being exercised early
* However, the equation to solve contains a maximum function - it cannot be directly solved
* Thus, there is a need to assume **different cases** so that we can eliminate certain maximum functions

$$S\_ 0 – k > e^{-rt}[p \cdot \max(K – S\_u, 0)]$$







$$\max(K – S\_u, 0) = K – S\_U$$

$$\max(K – S\_D, 0) = K – S\_D$$







$$\max(K – S\_U, 0) = 0$$

$$\max(K – S\_D, 0) = K – S\_D$$







$$\max(K – S\_U , 0) = 0$$

$$\max(K – S\_D, 0) = 0 $$



